Outline of talk

- Incoming and outgoing solutions

- Rellich's uniqueness theorem I: (*)
  V real valued, $\lambda \in \mathbb{R} \setminus \{0\} \Rightarrow$ no outgoing solutions

- Rellich's uniqueness theorem II:
  more general operators

- Conditions for outgoing solutions
  and decompositions of free plane waves

Sets us up for (next time) the scattering matrix in odd dimensions

(*) if time: more on Carleman estimates

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Incoming and outgoing solutions

Recall from 1D (sections 2.1, 2.4):

u solves $(P_V - \lambda^2)u = 0 \Rightarrow$ outside $\operatorname{supp} V$ $u(x) = u_{in}(x) + u_{out}(x)$

$u_{in}(x) = b_{\operatorname{sgn}(x)} e^{-i\lambda|x|}$ $\quad u_{out}(x) = a_{\operatorname{sgn}(x)} e^{i\lambda|x|}$

study map $S : \begin{pmatrix} b_- \\ b_+ \end{pmatrix} \to \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$ by finding solutions

$u^{\pm}(x) = e^{\pm i\lambda x} + \underbrace{v^{\pm}(x,\lambda)}_{\text{outgoing}}$

Def: A solution u to $(P_V - \lambda^2)u = f$, $\lambda \in \mathbb{R}\setminus\{0\}$, $f \in L^2_c(\mathbb{R}^n)$ is <u>outgoing</u> if $\exists g \in L^2_c(\mathbb{R}^n)$ such that $u = R_0(\lambda)g$, <u>incoming</u> if $u = R_0(-\lambda)g$

Look for w solves $(P_V - \lambda^2)w = 0$, $w(x,\lambda,\omega) = e^{-i\lambda\langle x,\omega\rangle} + u(x,\lambda,\omega)$, u outgoing. We have $u(x,\lambda,\omega) = -R_V(\lambda)(Ve^{-i\lambda\langle\cdot,\omega\rangle})$.

Asymptotic expansions from §3.1:

$$u(x) = \frac{e^{\pm i\lambda|x|}}{|x|^{\frac{n-1}{2}}}\, a\left(\frac{x}{|x|}\right) + O\left(|x|^{-\left(\frac{n+1}{2}\right)}\right) \text{ as } |x| \to \infty$$

intensity in direction $\omega \in S^{n-1}$

$+$ = outgoing, $-$ = incoming

②

The formula $u(x,\lambda,w) = -R_V(\lambda)(Ve^{-i\lambda\langle\cdot,w\rangle})$ makes sense if λ is not a pole

Rellich's uniqueness theorem I

__Thm:__ Suppose $V \in L^\infty_c(\mathbb{R}^n;\mathbb{R})$. Then for $\lambda \in \mathbb{R}\setminus\{0\}$,
$\nexists$ outgoing solns to $(P_V - \lambda^2)u = 0 \Leftrightarrow R_V(\lambda)$ has no poles $\lambda \in \mathbb{R}\setminus\{0\}$

__pf:__ (a) R_V has a pole $\Rightarrow$ outgoing solution

R_V has a pole $\Rightarrow I + VR_0(\lambda)\rho$ not invertible

$\Rightarrow \exists\, g = \rho g \in L^2_c$ with $g = -VR_0(\lambda)\rho g$

$\Rightarrow u = R_0(\lambda)g$ solves $(P_V - \lambda^2)u = 0$

(b) Outgoing soln $\Rightarrow R_V$ has a pole

$u = R_0(\lambda)g$, $(P_V - \lambda^2)u = 0 \Rightarrow (I + VR_0(\lambda)\rho)g = 0$

$\Rightarrow m_H(\lambda) > 0$

$\Rightarrow m_R(\lambda) > 0$

Recall from last time: $H(\lambda) = \sum_{j=0}^{n}(-VR_0(\lambda))^j(I + VR_0(\lambda)\rho)$, $m_R(\lambda) \le m_H(\lambda)$

Now main statement. Contradiction: Suppose $(P_V - \lambda^2)w = 0$, $w = -R_0(\lambda)Vw$

3 steps:

1) $w = O(r^{-\frac{n+1}{2}})$
2) w compactly supported
3) $w \equiv 0$

(3)

1) asymptotics for $R_0(\lambda)$ show

$$w = \frac{e^{i\lambda|x|}}{|x|^{\frac{n-1}{2}}}\left(h\left(\frac{x}{|x|}\right) + O\left(\frac{1}{|x|}\right)\right),$$

$$h(\theta) = c_n \lambda^{\frac{n-3}{2}} \widehat{Vw}(\lambda\theta)$$

so $(\partial_r - i\lambda)w = O(r^{-(\frac{n+1}{2})})$ $(*)$

$\lambda \in \mathbb{R}$, so

$$0 = \int_{B(0,R)} (w(P_V - \lambda^2)\bar{w} - (P_V - \lambda^2)w\bar{w})\,dx$$

$$= \int_{B(0,R)} (\bar{w}\Delta w - w\Delta\bar{w})\,dx \overset{IBP}{=} \int_{\partial B(0,R)} (\partial_r w\bar{w} - w\partial_r\bar{w})\,dS$$

using $(*)$, $0 = \int_{\partial B(0,R)} (i\lambda w + O(r^{-(\frac{n+1}{2})}))(\underset{\bar{w}}{O}(r^{-(\frac{n-1}{2})})) - (-i\lambda\bar{w} + O(r^{-(\frac{n+1}{2})}))(\underset{w}{O}(r^{-(\frac{n-1}{2})}))$

$\Rightarrow$ $0 = 2i\lambda\int_{\partial B(0,R)} |w|^2 dS + O(R^{-1})$

$\Rightarrow$ $\int_{\partial B(0,R)} |w|^2 dS = O(R^{-1})$

$\Rightarrow$ $w = O(R^{-(\frac{n+1}{2})})$

(4)

2) from 1), know $\widehat{Vw}(\xi) \equiv 0$ on sphere $\langle \xi, \xi \rangle = \lambda^2 \subset \mathbb{R}^n$

$\Sigma = \{ \xi \in \mathbb{C}^n : \langle \xi, \xi \rangle = \lambda^2 \}$ is connected, $\widehat{Vw}(\xi)$ is entire (since Vw compactly supported) + vanishes on $\Sigma \cap \mathbb{R}^n$ so it vanishes on Σ.

$$\frac{\widehat{Vw}(\xi)}{\langle \xi, \xi \rangle - \lambda^2} \text{ is entire on } \mathbb{C}^n$$

$$(\langle \xi, \xi \rangle - \lambda^2) \hat{w}(\xi) = \widehat{Vw}(\xi) \quad \text{by assumption}$$

$\overset{\text{Paley-Wiener}}{\Longrightarrow}$ w has compact support.

3) uses a Carleman estimate

Lemma: $\forall R > 0$, $\exists \varphi \in C^\infty(\mathbb{R}^n; \mathbb{R})$ s.t. $\forall h > 0$, $u \in H^2(\mathbb{R}^n)$, $\operatorname{supp} u \subset B(0,R)$, we have

$$\| h^2 e^{\varphi/h} \Delta e^{-\varphi/h} u \|_{L^2} \geq c h^{\frac{1}{2}} \|u\|_{L^2}$$

Carleman estimates are useful for unique continuation problems, proof and more on them later

⑤

let $u = e^{\varphi/h} w$, $w \in H^2$, $\operatorname{supp} w \in B(0,R)$

$$0 = h^2 \| e^{\varphi/h} (P_V - \lambda^2) w \|_{L^2}$$

$$= \| e^{\varphi/h} (-h^2 \Delta + h^2 V - h^2 \lambda^2) e^{-\varphi/h} u \|_{L^2}$$

$$\geq \| e^{\varphi/h} (-h^2 \Delta) e^{-\varphi/h} u \|_{L^2} - Ch^2 \|u\|_{L^2}$$

$$\geq ch^{\frac{1}{2}} \|u\|_{L^2} - Ch^2 \|u\|_{L^2} \geq \left(\frac{c}{2}\right) h^{\frac{1}{2}} \|u\|_{L^2}$$

for h small enough, so $u \equiv 0 \Rightarrow w \equiv 0$

$\square$

Rellich's uniqueness theorem II

Thm: Suppose P is a self-adjoint operator with domain $H^2(\mathbb{R}^n)$ such that for $\chi \in C_c^\infty(B(0,R))$, $\chi \equiv 1$ in $B(0,R)$, we have $P(1-\chi) = -\Delta(1-\chi)$.
Suppose $\lambda > 0$, $u \in H^2_{loc}$ with

$$(**) \quad (P - \lambda^2) u = 0, \quad \lim_{R \to \infty} \int_{\partial B(0,R)} |\partial_r - i\lambda) u|^2 \, dS = 0$$

Then $u \equiv 0$ for $|x| > R$

(6)

Remarks:

- P only needs to look like $-\Delta$ asymptotically, be self-adjoint
- (**) is implied by $(\partial_r - i\lambda)u = o(r^{-\frac{n-1}{2}})$, the Sommerfeld radiation condition.

name comes from physics; idea is that radiation should escape to infinity not come in from infinity; rules out unphysical solutions

pf sketch: (similar approach to earlier thm)

$$(-\Delta-\lambda^2)(1-\chi)u = [\Delta,\chi]u =: f \in C_c^\infty(\mathbb{R}^n)$$

claim: $(1-\chi)u = R_0(\lambda)f$.

define $w = (1-\chi)u - R_0(\lambda)f$, w solves $(-\Delta-\lambda^2)w=0$

for $G = \frac{1}{2i\lambda}(\partial_r - i\lambda)w$, argue as in 1) above using integration by parts that

$$0 \geq \frac{1}{2}\int_{B(0,R)} |w|^2 ds - 2\int_{\partial B(0,R)} |G|^2 dS,$$ use (**)

to conclude $\frac{1}{R}\int_{B(0,R)} |w|^2 dx \to 0$ as $R \to \infty$

$\operatorname{supp} \hat{w} \subset \{|\xi|^2 = \lambda^2\}$, apply distribution theory lemma $\Rightarrow w=0$

7 having shown $(1-\chi)u = R_0(\lambda) f$, $r = [\Delta, \chi] u$, for $\chi_1 \in C_c^\infty(\mathbb{R}^n)$ equal to 1 on $\operatorname{supp} \chi$, asymptotics for R_0 show

$$(\dagger) \quad \frac{1}{i} \langle [-\Delta, \chi_1] R_0(\lambda) f, R_0(\lambda) f \rangle = |c_n|^2 \lambda^{n-2} \int_{S^{n-1}} |\hat{f}(\lambda\theta)|^2 d\theta$$

can also show $\langle [-\Delta, \chi_1] R_0(\lambda) f, R_0(\lambda) f \rangle = 0$,

so $\hat{f}(\lambda\theta) \equiv 0$ for $\theta \in S^{n-1}$, argue as in step 2) above. $\square$

the expression on the LHS of $(\dagger)$ is called quantum flux

is positive for outgoing solutions, negative for incoming.

More on outgoing solutions

Thm: Suppose $V \in L_c^\infty(\mathbb{R}^n; \mathbb{R})$, $f \in \mathcal{E}'(\mathbb{R}^n)$ (compactly supported distributions),

$(P_V - \lambda^2) u = f$, $\lambda \in \mathbb{R} \setminus \{0\}$.

The following are equivalent.

i) $u(x) = e^{i\lambda|x|} a\left(\frac{x}{|x|}\right) |x|^{-\frac{(n-1)}{2}} + O\left(|x|^{-\frac{(n+1)}{2}}\right)$ as $x \to \infty$

ii) $(\partial_r - i\lambda) u = o\left(r^{-(n-1)/2}\right)$, $r = |x|$

iii) $u = R_V(\lambda) f$

iv) $u = R_0(\lambda) g$, $g \in \mathcal{E}'$ (when $V \in C_c^\infty$, $f \in C_c^\infty$, then $g \in C_c^\infty$)

pf: Exercise 3.5

(8)

Want to decompose $w(x,\lambda,\omega) = e^{-i\lambda\langle x,\omega\rangle} + u(x,\lambda,\omega)$ into incoming + outgoing terms (then define scattering matrix)

<u>Thm</u>: for $\lambda \in \mathbb{R}\setminus\{0\}$, in the sense of distributions in $\frac{x}{|x|} \in S^{n-1}$

$$e^{-i\lambda\langle x,\omega\rangle} \sim \frac{1}{(\lambda|x|)^{\frac{n-1}{2}}}\left(c_n^+ e^{-i\lambda|x|}\delta_\omega(x/|x|) + c_n^- e^{i\lambda|x|}\delta_{-\omega}(x/|x|)\right)$$

as $|x| \to \infty$, with $c_n^{\pm} = (2\pi)^{\frac{n-1}{2}} e^{\pm\frac{\pi}{4}(n-1)i}$

Remarks: - $\lambda^{-\frac{n-1}{2}} c_n^{\pm} \delta_{\pm\omega}(\theta)$ can be thought of as leading coefficients of incoming (+) and outgoing (−) components of $e^{-i\lambda\langle x,\omega\rangle}$

- when paired with $\varphi \in C^\infty(S^{n-1})$, remainder is $O(\frac{1}{r})$ as $r \to \infty$, full expansion in powers of r possible

- Compare 1-D case: $e^{\pm i\lambda x} = e^{-i\lambda|x|}(\pm x)_-^0 + e^{i\lambda|x|}(\pm x)_+^0$

Recall the <u>method of stationary phase</u> for the oscillatory integral

$$\int_{S^{n-1}} e^{-i\lambda r\langle\omega,\theta\rangle}\varphi(\theta)\,d\theta$$

idea: main contribution comes near critical points of phase

9 Brief overview of stationary phase:

We have an oscillatory integral

$$\int_{\mathbb{R}^n} e^{i\varphi(x)/h} a(x)\,dx =: I_h \qquad \begin{array}{l}\varphi(x) \in C^\infty(\mathbb{R}^n \to \mathbb{R}) \\ a(x) \in C_c^\infty(\mathbb{R}^n)\end{array}$$

Nonstationary phase: If φ has no critical points on supp a,

consider $L := \frac{h}{i} \frac{1}{|\nabla\varphi|^2} \nabla\varphi \cdot \nabla$

$$L(e^{i\varphi(x)/h}) = e^{i\varphi(x)/h}$$

integrate by parts N times: $\int_{\mathbb{R}^n} e^{i\varphi(x)/h} a(x)\,dx = \int_{\mathbb{R}^n} L^N(e^{i\varphi(x)/h}) a(x)\,dx$

$$= \int_{\mathbb{R}^n} e^{i\varphi(x)/h} (L^*)^N a(x) \le C_a h^N$$

Stationary phase: If $\nabla\varphi(x_0) = 0$, $\nabla^2\varphi(x_0)$ nonsingular,

φ has no further critical points near x_0, then

$$I_h = (2\pi h)^{n/2} |\det \nabla^2\varphi(x_0)|^{-1/2} e^{i\frac{\pi}{4}\operatorname{sgn}(\nabla^2\varphi(x_0))} e^{i\frac{\varphi(x_0)}{h}} a(x_0)$$

$$+ O(h^{\frac{n+2}{2}}),$$ full expansion possible

idea of proof: convert φ to a quadratic near x_0 via Morse Lemma

(10)

pf of thm: WLOG $w = (1, 0, \dots, 0)$

$\langle w, \theta \rangle$ on S^{n-1} has critical points at $\pm(1, 0, \dots, 0)$

1) By stationary phase, contributions far from $(1, 0, \dots, 0)$ are $O((\lambda r)^{-\infty})$

2) near poles, write $\theta = (\pm\sqrt{1-|t|^2}, t)$

and we have

$$\int_{S^{n-1}} e^{-i\lambda r\langle w, \theta\rangle} \varphi(\theta)\, d\theta = \int_{B_{\mathbb{R}^{n-1}}(0,1)} e^{\mp \lambda r\sqrt{1-|t|^2}} \varphi(\pm\sqrt{1-|t|^2}, t) J(t)\, dt$$

J: Jacobian, is $1 + O(t^2)$

$\nabla^2 (\mp\lambda r\sqrt{1-|t|^2})\big|_{t=0} = \pm I_{\mathbb{R}^{n-1}}$, so stationary phase

gives

$$\int_{B_{\mathbb{R}^{n-1}}(0,1)} e^{\mp i\lambda r\sqrt{1-|t|^2}} \varphi(\pm\sqrt{1-|t|^2}, t) J(t)\, dt \sim$$

$$\left(\frac{2\pi}{r\lambda}\right)^{\frac{n-1}{2}} e^{\pm i\frac{\pi}{4}(n-1)} \left(\varphi(\pm 1, 0) + O\left(\frac{1}{r\lambda}\right)\right)$$

full expansion in powers of $r\lambda$ is possible

3) can write $\underset{\substack{\text{supp near}\\ \pm(1,0,\dots,0)}}{\varphi = \varphi_1} + \underset{\substack{\text{supp far from}\\ \pm(1,0,\dots,0)}}{\varphi_2}$,

So we are done. $\square$

(11)

remark: the full stationary phase asymptotics are important for proving trace formulas later.

One last result (to help set up for next time)

Thm: Let P be self-adjoint with domain $H^2(\mathbb{R}^n)$ with $P(1-\chi) = -\Delta(1-\chi)$ for $\chi \in C_c^\infty(B(0,2R);\mathbb{R})$ identically 1 in $B(0,R)$.

Let f_ℓ, $\ell = 1,2$ be Schwartz functions, $f_\ell, g_\ell \in C^\infty(S^{n-1})$

$\lambda \in \mathbb{R} \setminus \{0\}$

$$(P-\lambda^2)u_\ell = F_\ell \,, \quad u_\ell(r\theta) = r^{-\frac{n-1}{2}}\left(e^{i\lambda r} f_\ell(\theta) + e^{-i\lambda r} g_\ell(\theta) + O(r^{-(\frac{n+1}{2})})\right)$$

Then

$$2i\lambda \int_{S^{n-1}} (g_1\bar{g}_2 - f_1\bar{f}_2)\, d\omega = \int_{\mathbb{R}^n} (F_1\bar{u}_2 - u_1\bar{F}_2)\, dx$$

pf sketch: use self-adjointness of P, definitions of F_1, F_2 on $B(0,r)$, integrate by parts, use expansions for u_1, u_2, take limit as $r \to \infty$

(12)

More on Carleman estimates

Lemma: $\forall R>0, \exists \varphi \in C^\infty(\mathbb{R}^n;\mathbb{R})$ s.t. $\forall h>0, u \in H^2(\mathbb{R}^n)$, $\operatorname{supp} u \subset B(0,R)$, we have

$$\|h^2 e^{\varphi/h} \Delta e^{-\varphi/h} u\|_{L^2} \geq c h^{1/2} \|u\|_{L^2}$$

pf: Let $P_\varphi = -h^2 e^{\varphi/h} \Delta e^{-\varphi/h}$

(the conjugated operator)

in general, symbol looks like $p(x, D - i\frac{\nabla\varphi}{h})$

$$\|P_\varphi u\|_{L^2}^2 = \langle P_\varphi^* P_\varphi u, u\rangle$$

$$= \|P_\varphi^* u\|_{L^2}^2 + \langle [P_\varphi^*, P_\varphi]u, u\rangle$$

$$\geq \langle [P_\varphi^*, P_\varphi]u, u\rangle$$

Want φ s.t. $\langle [P_\varphi^*, P_\varphi]u, u\rangle \geq ch\|u\|_{L^2}^2$

$$P_\varphi u = -h^2\Delta u + 2h\, \nabla\varphi\cdot\nabla u - |\nabla\varphi|^2 u + h(\Delta\varphi)u$$

$$P_\varphi^* u = -h^2\Delta u - 2h\,\nabla\varphi\cdot\nabla u - |\nabla\varphi|^2 u - h(\Delta\varphi)u$$

$$[P_\varphi^*, P_\varphi]u = -8h^3\sum_{j,k=1}^{n} \partial^2_{x_j x_k}\varphi\, \partial^2_{x_j x_k} u + 4h\,\nabla\varphi\cdot\nabla(|\nabla\varphi|^2)u$$

$$-8h^3(\nabla\Delta\varphi)\cdot\nabla u - 2h^3(\Delta^2\varphi)u$$

Pick $\varphi(x) = |x|^2/2 + Mx_1$, $M \geq R+1$, and since

$[P^*, P_\varphi]u = 8h(-h^2\Delta u + |x+Me_1|^2 u)$, we are done. □